On Subharmonic Functions.

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1. The following theorem has been given by MONTEL:

If $u(x, y)$ *is continuous and* $e^{u(x, y)+\alpha x+\beta y}$ *subharmonic for all values of* α, β, *then* $u(x, y)$ *is subharmonic.*

MONTEL[1]) proved this theorem under the assumption that $u(x, y)$ has continuous partial derivatives of the first and second order. The above general statement was proved by RADÓ.[2])

KIERST has recently found the following generalization of the theorem of MONTEL.

If $u(x, y)$ *and* $f(t)$ *have continuous derivatives of the first and second order, if* $f'(t)$ *is positive and if* $f(u+\alpha x+\beta y)$ *is subharmonic for all values of* α, β, *then* $u(x, y)$ *is subharmonic.*

The proof is almost identical to the argument used by MONTEL. We have, $f(u+\alpha x+\beta y)$ being subharmonic,

$$\Delta f(u+\alpha x+\beta y) = f''(u+\alpha x+\beta y)\left[(u'_x+\alpha)^2+(u'_y+\beta)^2\right] + \\ + f'(u+\alpha x+\beta y)\, \Delta u \geqq 0.$$

Give x and y any fixed values and put

$$\alpha = -u'_x(x, y), \quad \beta = -u'_y(x, y).$$

We get then $\Delta u \geqq 0$ for all values of x, y. This proves the subharmonic character of $u(x, y)$.

2. RADÓ remarked, in conversations with the author, that if the linear function $\alpha x+\beta y+\gamma$ with three parameters is introduced instead of $\alpha x+\beta y$, the assumption $f'(t)>0$ can be dropped.

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1) P. MONTEL, Sur les fonctions convexes et les fonctions sousharmoniques, *Journal de Mathématiques*, (9) **7** (1928), p. 29—60, especially p. 40.

2) T. RADÓ, Remarque sur les fonctions subharmoniques, *Comptes Rendus, Paris,* **186** (1928), p. 346—348.

We have then the theorem: if $f(t)$ and $u(x, y)$ have continuous derivatives of the first and second order, if $f(t)$ is not equal identically to a constant and if $f(u+\alpha x+\beta y+\gamma)$ is subharmonic for all values of α, β, γ, then $u(x, y)$ is either subharmonic or superharmonic.

Indeed, because $f(t)$ is not constant, there exists a t_0 such that $f'(t_0) \neq 0$. Suppose, for instance,

$$f'(t_0) > 0. \tag{1}$$

For any fixed values of x, y we can put then

$$\alpha = -u'_x(x, y),\ \beta = -u'_y(x, y),\ \gamma = t_0 - \alpha x - \beta y - u(x, y)$$

and from the relation

$$\begin{aligned}(2)\quad & \Delta f(u+\alpha x+\beta y+\gamma) = \\ & = f''(u+\alpha x+\beta y+\gamma)[(u'_x+\alpha)^2+(u'_y+\beta)^2]+f'(u+\alpha x+\beta y+\gamma)\Delta u \geqq 0\end{aligned}$$

which expresses the subharmonic character of $f(u+\alpha x+\beta y+\gamma)$, we obtain by the relation (1)

$$\Delta u \geqq 0$$

which proves that $u(x, y)$ is subharmonic.

The assumption concerning the regularity of $f(t)$ can be easily dropped by using the method of the integral means (see § 4). However, this method does not allow us to get rid of the restriction concerning the regularity of $u(x, y)$ and we shall use a different method for this purpose.

3. We shall prove by this method the following theorem.

If $f(t)$ is continuous for all values of t, $u(x, y)$ continuous in an open plane region R, and if $f(u+\alpha x+\beta y+\gamma)$ is subharmonic in R for all values of α, β, γ, then

1. *$f(t)$ is a convex function, and*

2. *either $f(t)$ is constant, or $u(x, y)$ is harmonic, or $f(t)$ is not-decreasing and $u(x, y)$ subharmonic, or $f(t)$ is not-increasing and $u(x, y)$ superharmonic.*

If $f(t) = e^t$, we have the theorem of MONTEL as completed by RADÓ.

The third parameter γ is essential for our proof. It is likely, however, that γ is not necessary for the validity of the theorem itself, at least if $f(t)$ is supposed to be monotonic. The investigation of the true role of γ might be, so it seems to the author, of great interest.

4. We first prove that under the assumptions of the preceding paragraph $f(t)$ is a *convex function.*

Suppose that $f(t)$ is not convex. Then there exists an interval $a \leq t \leq b$ and a linear function $mt+n$ such that the function

$$F(t) = f(t) + mt + n \tag{1}$$

vanishes for a, b and takes on a positive maximum for some value c in (a, b). Put

$$F(c) = M \tag{2}$$

and denote by η a positive number such that

$$|f(t+h) - f(t)| < \frac{M}{4} \tag{3}$$

for every t in the interval (a, b) and for every $|h| \leq \eta$.

On account of the parameter γ in $f(u+\alpha x+\beta y+\gamma)$, it can obviously be supposed that the region R, where $u(x, y)$ is defined, contains the origin and that $u(0, 0) = 0$.[3]) Let r be a positive number such that $u(x, y) \leq \eta$ for $|x|, |y| \leq r$. We have then by (1), (2), (3) for every $|x|, |y| \leq r$

$$\left.\begin{array}{l} f[u(x,y)+a]+ma+n \\ f[u(x,y)+b]+mb+n \end{array}\right\} < \frac{M}{4} < \frac{3M}{4} < f[u(x,y)+c]+mc+n \tag{4}$$

and for every $a \leq \xi \leq b$

$$f[u(x,y)+\xi]+m\xi+n < \frac{5M}{4} < f[u(x,y)+c]+mc+n+\frac{M}{2}. \tag{5}$$

Choose now $\alpha > 0$ so as to have

$$\left|\frac{a}{\alpha}\right|, \left|\frac{b}{\alpha}\right|, \left|\frac{c}{\alpha}\right| < r$$

and put

$$\Phi(x, y) = f[u(x, y) \mp \alpha x] + f[u(x, y) + \alpha y] + m\alpha(x+y).$$

It follows from (4) and (5) that

$$\Phi\left(\frac{c}{\alpha}, \frac{c}{\alpha}\right) > \begin{cases} \Phi\left(\dfrac{a}{\alpha}, y\right), \Phi\left(\dfrac{b}{\alpha}, y\right), \text{ for } \dfrac{a}{\alpha} < y < \dfrac{b}{\alpha}, \\ \Phi\left(x, \dfrac{a}{\alpha}\right), \Phi\left(x, \dfrac{b}{\alpha}\right), \text{ for } \dfrac{a}{\alpha} < x < \dfrac{b}{\alpha}. \end{cases}$$

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3) It may be observed that if the region R contains the origin, the third parameter γ becomes superfluous for the proof of the convexity of $f(t)$ (cf. § **3**). Neither is the assumption $u(0, 0) = 0$ necessary. The parameter γ is also superfluous if we suppose that $f(t)$ and $u(x, y)$ have continuous first and second derivatives, and in this case the proof can be obtained by the same easy computations as in § **1** and § **2**.

This means that $\Phi(x,y)$ takes on its maximum for the square $\left(\frac{a}{\alpha}, \frac{a}{\alpha}; \frac{b}{\alpha}, \frac{b}{\alpha}\right)$ in some interior point of the square. This however is impossible because Φ is obviously subharmonic.

5. We shall use in the sequel a slight generalization of the following theorem of Blaschke.[4])

In order that a continuous function $U(x,y)$ be harmonic in an open region D, it is necessary and sufficient that

$$\lim_{r\to 0} \frac{1}{2\pi r^2}\int_0^{2\pi}[U(x_0+r\cos\varphi,\ y_0+r\sin\varphi)-U(x_0,y_0)]\,d\varphi=0$$

for every point (x,y) of D.

It is immediate that this theorem can be generalized as follows.[5])

In order that a continuous function $U(x,y)$ be subharmonic in an open region D, it is necessary and sufficient that

$$(1)\qquad \limsup_{r\to 0} \frac{1}{2\pi r^2}\int_0^{2\pi}[U(x+r\cos\varphi,\ y+r\sin\varphi)-U(x,y)]\,d\varphi\geqq 0$$

for every point x, y of D.

The proof is based on exactly the same method as that of Blaschke. Let n be a positive integer and put

$$U_n(x,y)=U(x,y)+\frac{x^2}{n}.$$

We have then by (1)

$$\limsup_{r\to 0} \frac{1}{2\pi r^2}\int_0^{2\pi}[U_n(x+r\cos\varphi,\ y+r\sin\varphi)-U_n(x,y)]\,d\varphi\geqq\frac{1}{2n}.$$

It follows from this inequality that to every point (x,y) of D there corresponds a sequence of values of r converging toward zero and such that

$$U_n(x,y)<\int_0^{2\pi}U_n(x+r\cos\varphi,\ y+r\sin\varphi)\,d\varphi.$$

[4]) W. Blaschke, Ein Mittelwertsatz und eine kennzeichnende Eigenschaft des logarithmischen Potentials, *Leipziger Berichte*, **68** (1916), p. 3—7.

[5]) S. Saks, On Convex and Subharmonic Functions (in Polish), *Mathesis Polska*, **6** (1931), p. 43—66.

This shows that U_n is subharmonic.[6] As U is, for $n \to \infty$, the uniform limit of U_n, it follows that U is also subharmonic.

6. We shall prove the following lemma.

If $f(t)$ is continuous for all values of t and $u(x, y)$ is continuous in an open region R, and if $f(u + \alpha x + \beta y + \gamma)$ is subharmonic for all values of α, β, γ, then $f(u + v)$ is subharmonic for every harmonic function $v(x, y)$.

1. Suppose first that $f(t)$ has a continuous first differential coefficient. Let (x_0, y_0) be a point of R. Put

$$x = x_0 + r\cos\varphi,\ y = y_0 + r\sin\varphi.$$

As v is harmonic, we have for small values of r an expansion

$$v(x, y) = v(x_0 + r\cos\varphi,\ y_0 + r\sin\varphi) =$$
$$= v(x_0, y_0) + \sum_{n=1}^{\infty} r^n (a_n \cos n\varphi + b_n \sin n\varphi).$$

Put

$$g(x, y) = f[u(x, y) + v(x, y)],$$
$$h(x, y) = f[u(x, y) + v(x_0, y_0) + a_1(x - x_0) + b_1(y - y_0)].$$

Denote by A the value of $f'(t)$ for $t = u(x_0, y_0) + v(x_0, y_0)$. We have then

$$(1)\left\{\begin{aligned} &\int_0^{2\pi} [g(x_0 + r\cos\varphi,\ y_0 + r\sin\varphi) - h(x_0 + r\cos\varphi,\ y_0 + r\sin\varphi)]\, d\varphi = \\ &= \int_0^{2\pi} \left\{ A \sum_{n=2}^{\infty} r^n (a_n \cos n\varphi + b_n \sin n\varphi) + r^2 \varepsilon(r, \varphi) \right\} d\varphi = \\ &= 2\pi r^2 \varepsilon(r, \varphi), \end{aligned}\right.$$

where $\varepsilon(r, \varphi)$ converges uniformly toward zero for $r \to 0$. As $h(x, y)$ is subharmonic by assumption, we have for sufficiently small values of r

$$\int_0^{2\pi} [h(x_0 + r\cos\varphi,\ y_0 + r\sin\varphi) - h(x_0, y_0)]\, d\varphi \geqq 0$$

and by (1)

$$\liminf_{r \to 0} \frac{1}{2\pi r^2} \int_0^{2\pi} [g(x_0 + r\cos\varphi,\ y_0 + r\sin\varphi) - g(x_0, y_0)]\, d\varphi \geqq$$

$$\geqq \liminf_{r \to 0} \frac{1}{2\pi r^2} \int_0^{2\pi} [g(x_0 + r\cos\varphi,\ y_0 + r\sin\varphi) -$$
$$- h(x_0 + r\cos\varphi,\ y_0 + r\sin\varphi)]\, d\varphi \geqq 0.$$

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[6] See for instance J. E. LITTLEWOOD, On the definition of a subharmonic function, *The Journal of the London Math. Society*, **2** (1927), p. 189—192.

On account of the theorem of the preceding section, this means that $g(x, y)$ is subharmonic.

2. We supposed above that f has a continuous first differential coefficient. We can get rid of this assumption by a method suggested by RADÓ and used by him in similar cases.[7]) Suppose that $f(t)$ is continuous and that $f(u+\alpha x+\beta y+\gamma)$ is subharmonic for all values of α, β, γ, It can be shown readily that the function

$$F_\delta(t)=\frac{1}{\delta}\int_0^\delta f(t+\vartheta)\,d\vartheta \qquad (\delta>0)$$

has the same property, that is to say that $F_\delta(u+\alpha x+\beta y+\gamma)$ is again subharmonic for all values of α, β, γ. As $F_\delta(t)$ has the continuous differential coefficient

$$F'_\delta(t)=\frac{1}{\delta}\,[f(t+\delta)-f(t)],$$

it follows from 1. that $F_\delta(u+v)$ is subharmonic for every $\delta>0$ and every harmonic function v, and that consequently

$$f(u+v)=\lim_{\delta\to 0} F_\delta(u+v)$$

is subharmonic.

7. We are now going to make the last step of our proof. We shall show, under the assumptions of **§ 3,** that *if $u(x, y)$ is not subharmonic, then $f(t)$ is necessarily monotonic and not-increasing and that if $u(x, y)$ is not superharmonic, then $f(t)$ is monotonic and not-decreasing.*

Suppose u is not subharmonic. We shall prove that then the right-hand derivative $f'_+(t)$ which exists on account of the convexity of $f(t)$ is everywhere $\leqq 0$.

Indeed, suppose that at some point a

$$f'_+(a)>0.$$

Then there exists a positive ε such that

(1) $\qquad f(a)<f(x)$, if $\quad 0<x-a<\varepsilon.$

$u(x, y)$ being not subharmonic, there exists a point (x_0, y_0) in R such that

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[7]) The whole theorem of this section could also be proved directly by using one-sided derivatives of $f(t)$. As $f(t)$ is convex, these derivatives exist and are monotonic and not decreasing.

$$(2) \qquad u(x_0, y_0) > \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\cos\varphi, y_0 + r\sin\varphi)\, d\varphi$$

for small r. We can choose r so small that the oscillation of $u(x, y)$ in the circle $(x-x_0)^2 + (y-y_0)^2 \leq r^2$ be less than $\frac{\varepsilon}{2}$.

Denote now by $v(x, y)$ the function which is continuous in the closed circle $(x-x_0)^2 + (y-y_0)^2 \leq r^2$, harmonic in the interior of this circle and equal to $u(x, y)$ on its boundary. The oscillation of v in this circle is then obviously less than $\frac{\varepsilon}{2}$. Put

$$w(x, y) = u(x, y) - v(x, y) + a.$$

This function has the constant value a on the boundary of the circle and in virtue of (2) it takes on a value larger than a at the center (x_0, y_0). As the oscillation of w in the circle is less than ε, we get from (1)

$$f[w(x_0, y_0)] > f(a) = \frac{1}{2\pi} \int_0^{2\pi} f[w(x_0 + r\cos\varphi, y_0 + r\sin\varphi)]\, d\varphi.$$

But this is impossible because the function

$$f[w(x, y)] = f(u - v + a)$$

is subharmonic by the lemma of the preceding section.

We prove in the same manner that if $u(x, y)$ is not superharmonic, then the function $f(t)$ has a non-negative left-hand derivative, that is to say it is not-decreasing.

The second part of the theorem in **§ 3** follows now immediately: if $u(x, y)$ is neither subharmonic nor superharmonic, then $f(t)$ is monotonic and both not-increasing and not-decreasing, that is to say $f(t)$ reduces to a constant identically.

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